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## Vector spaces

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**Definition:** A **vector space** is a nonempty set  $V$  of objects, called *vectors*, on which two operations are defined: *addition* and *multiplication by scalars* (real numbers) which obey the following ten axioms. The axioms hold of all  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  in  $V$  and all scalars  $c$  and  $d$ .

1. The sum of  $\mathbf{u}$  and  $\mathbf{v}$ , denoted  $\mathbf{u} + \mathbf{v}$ , is in  $V$ .
2.  $\mathbf{u} + \mathbf{v} = \mathbf{v} + \mathbf{u}$ .
3.  $(\mathbf{u} + \mathbf{v}) + \mathbf{w} = \mathbf{u} + (\mathbf{v} + \mathbf{w})$ .
4. There is a **zero vector**  $\mathbf{0}$  in  $V$  such that  $\mathbf{u} + \mathbf{0} = \mathbf{u}$ .
5. For each  $\mathbf{u}$  in  $V$ , there is a vector  $-\mathbf{u}$  in  $V$  such that  $\mathbf{u} + (-\mathbf{u}) = \mathbf{0}$ .
6. The scalar multiple of  $\mathbf{u}$  by  $c$ , denoted  $c\mathbf{u}$ , is in  $V$ .
7.  $c(\mathbf{u} + \mathbf{v}) = c\mathbf{u} + c\mathbf{v}$ .
8.  $(c + d)\mathbf{u} = c\mathbf{u} + d\mathbf{u}$ .
9.  $c(d\mathbf{u}) = (cd)\mathbf{u}$ .
10.  $1\mathbf{u} = \mathbf{u}$ .

**Example 1:** The space  $\mathbb{R}^n$  for  $n \geq 1$  is the fundamental example of a vector space. We rely on the geometric intuition developed here to understand and visualize many concepts of general vector spaces.

**Example 2:** Let  $V$  be the set of all arrows (directed line segments) in three-dimensional space, with two arrows regarded as equal if they have the same length and point in the same direction. Define addition by the parallelogram rule (from section 1.3), and for each  $v$  in  $V$ , define  $cv$  to be the arrow whose length is  $|c|$  times the length of  $v$ , pointing in the same direction as  $v$  if  $c \geq 0$  and otherwise pointing in the opposite direction. Notice  $V$  is a vector space.

Verify that  $V$  satisfies axioms 1 and 2.

**Example 3:** Let  $M_{m \times n}$  be the set of all  $m \times n$  matrices with entries in  $\mathbb{R}$ . With addition and scalar multiplication defined as in section 2.1m  $M_{m \times n}$  is a vector space.

Verify that  $M_{m \times n}$  satisfies axioms 3 and 4.

**Example 4:** Let  $\mathbb{P}_n$  be the set of all polynomials of degree at most  $n$ . Addition is defined by combining like coefficients and scalar multiplication by scaling each coefficient. Note  $\mathbb{P}_n$  is a vector space.

Verify that  $\mathbb{P}_n$  satisfies axioms 5 and 6.

**Example 5:** Let  $W$  be the set of all real-valued functions on  $\mathbb{R}$ . For two such functions  $f$  and  $g$ , define addition by  $(f + g)(x) = f(x) + g(x)$  and scalar multiplication by  $(cf)(x) = c(f(x))$ . Then  $W$  is a vector space.

Verify that  $W$  satisfies axioms 7 and 8.

**Example 6:** Let  $\mathbb{S}$  be the set of all doubly infinite sequences of numbers

$$\{y_k\} = (\dots, y_{-2}, y_{-1}, y_0, y_1, y_2, \dots).$$

Define addition of two such sequences  $\{y_k\} + \{z_k\}$  as  $\{y_k + z_k\}$ . Define scalar multiplication as  $c\{y_k\} = \{cy_k\}$ . Then  $\mathbb{S}$  is a vector space.

Verify that  $\mathbb{S}$  satisfies axioms 9 and 10.

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## Subspaces

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**Definition:** A **subspace** of a vector space  $V$  is a subset  $H$  of  $V$  that has three properties

1. The zero vector of  $V$  is in  $H$ .
2.  $H$  is closed under vector addition. (That is, if  $\mathbf{u}$  and  $\mathbf{v}$  are in  $H$ , then  $\mathbf{u} + \mathbf{v}$  is also in  $H$ .)
3.  $H$  is closed under multiplication by scalars. (That is, for each  $\mathbf{u}$  in  $H$  and each scalar  $c$ , the vector  $c\mathbf{u}$  is in  $H$ .)

**Example 1:** The set  $\{\mathbf{0}\}$  is a subspace of any vector space, called the **zero subspace**.

**Example 2:** Let  $H = \left\{ \begin{bmatrix} a & a+b \\ 0 & b \end{bmatrix} : a, b \text{ in } \mathbb{R} \right\}$ . Show that  $H$  is a subspace of  $M_{2 \times 2}$ .

**Example 3:** Show that  $H = \{f : \mathbb{R} \rightarrow \mathbb{R} : f \text{ is differentiable}\}$  is a subspace of  $W$ .

**Example 4:** Notice that  $\mathbb{R}^2$  is *not* a subspace of  $\mathbb{R}^3$ . However,  $H = \left\{ \begin{bmatrix} a \\ b \\ 0 \end{bmatrix} : a, b \text{ in } \mathbb{R} \right\}$  is a subspace of  $\mathbb{R}^3$ .

Verify this.

**Example 5:** Show that  $\mathbb{D} = \{(x, y) : x^2 + y^2 < 1\}$  is not a subspace of  $\mathbb{R}^2$ .

**Example 6:** Show that for any  $\mathbf{v}_1$  and  $\mathbf{v}_2$  in a vector space  $V$ , the set  $H = \text{Span}\{\mathbf{v}_1, \mathbf{v}_2\}$  is a subspace of  $V$ .

**Example 7:** Let  $H = \{(a - 3b, b - a, a, b) : a, b \text{ in } \mathbb{R}\}$ . Show that  $H$  is a subspace of  $\mathbb{R}^4$ . (*Hint:* Find a spanning set for  $H$ )

**Example 8:** Show  $H = \{at^2 + at + a : a \text{ in } \mathbb{R}\}$  is a subspace of  $\mathbb{P}_n$  for  $n \geq 2$ .